The Eilenberg-Mazur swindle

Lucien Grillet

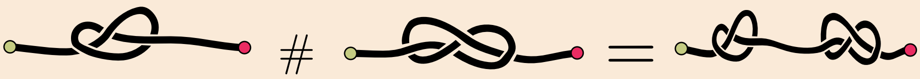
You may have already seen this "proof" showing that 1=0.

$$\begin{aligned}1 &= 1\\ &= 1+0+0+0+\dots\\ &= 1+(-1+1)+(-1+1)+(-1+1)+\dots\\ &= (1-1)+(1-1)+(1-1)+\dots\\ &= 0+0+0+\dots\\ &= 0\end{aligned}$$

Of course, this "proof" is **wrong** because this sum is not well defined.

But the idea used in this computation is actually **interesting**! What would happen if we tried to use it on objects that accept this kind of infinite sums?

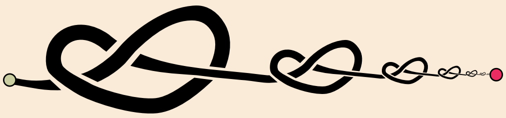
Think about knots between two points and the operation of concatenation:



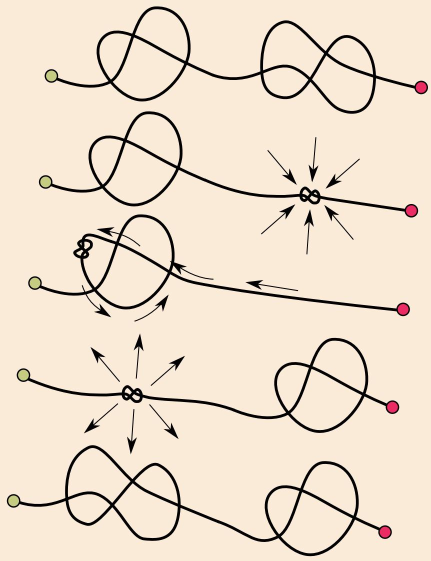
Two knots are considered equal if you can continuously move one to get the other without crossing (more rigorously, if there is an ambient homeomorphism sending one to the other).

For the "1=0 proof" to work, we need the operation # to be **associative** and **commutative**. We also need the **infinite concatenations** involved in the proof to be well defined.

We can create infinite concatenations of knots by making them smaller and smaller!



We can easily commute two knots.



With some thinking, one can also see that the operation of concatenation is associative and that the properties of commutativity and associativity remain true for infinite concatenations needed in the "1=0 proof".

So, what do we obtain if we apply the "1=0 proof" to this example? Do we obtain that every knot can be unknotted?
No! We obtain that for a given knot "1" there does not exist an inverse knot "-1" (Or, if it exists, the knot was unknotted to begin with).

In other words:

"To unknot a knot, one can not tie its anti-knot to the other end of the string and slide it until they both cancel out!"

The Eilenberg–Mazur swindle can be used to prove a large variety of **algebra** and **geometric topology** theorems.

Among them the **Poincaré conjecture in higher dimensons** can make use of this trick in its proofs. This theorem has a subtle statement on the connection between the topological and the smooth worlds.
One can see this infinite construction as the reason we only obtain an **homeomorphism** in this theorem, and not a **diffeomorphism**.

! Note that the infinite concatenation only works in the *topological world*. This proof does not work for smooth knots.